The quadratic formula

Our final task is to apply everything that we have learned to quadratic relations in standard form $y = ax^2 + bx + c$ for any coefficients a, b and c.

Example – Determine the axis of symmetry, vertex and x-intercepts of $y = 3x^2 + 5x + 1$

Complete the square: $y = 3\left[x^2 + \frac{5}{3}x \qquad \right] + 1$

$$y = 3\left[x^2 + \frac{5}{6}x + \frac{5}{6}x + \frac{25}{36} - \frac{25}{36}\right] + 1$$

$$y = 3\left[\left(x + \frac{5}{6}\right)^2 - \frac{25}{36}\right] + 1$$

axis of symmetry

$x = -\frac{5}{6}$

$$y = 3\left[x + \frac{5}{6}\right]^2 + 3\left(\frac{-25}{36}\right) + 1$$

vertex $\left(-\frac{5}{6}, \frac{-13}{12}\right)$

$$y = 3\left(x + \frac{5}{6}\right)^2 - \frac{25}{12} + \frac{12}{12}$$

$$y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$$

$$0 = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$$

$$\frac{13}{12} = 3\left(x + \frac{5}{6}\right)^2$$

x-intercepts

$$\frac{13}{36} = \left(x + \frac{5}{6}\right)^2$$

$(-0.23, 0)$

$$+0.60 \doteq x + \frac{5}{6} \quad \text{or} \quad -0.60 \doteq x + \frac{5}{6}$$

$(-1.43, 0)$

$$-0.23 \doteq x_1 \quad \text{or} \quad -1.43 \doteq x_2$$

Clearly, the process required to determine the x-intercepts of a quadratic relation from standard form is very tedious.

However, it will be the same process for every quadratic relation $y = ax^2 + bx + c$.

Therefore, we only need to complete the process once using a, b and c.

We can summarize the entire process into one formula for determining x-intercepts!

Our formula must accomplish two basic tasks:

1. Complete the square in order to convert from standard form into vertex form:

$$y = ax^2 + bx + c$$

$$y = a\left[x^2 + \frac{b}{a}x\right] + c$$

$$y = a\left[x^2 + \frac{b}{2a}x + \frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c$$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c\left(\frac{4a}{4a}\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{4ac}{4a}$$

when $x = -\frac{b}{2a}$ we can calculate/determine the maximum or minimum value of $y = ax^2 + bx + c$.

2. Apply opposite operations in order to determine the x-intercepts when y = 0:

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+\frac{4ac}{4a}=0$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-\frac{4ac}{4a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

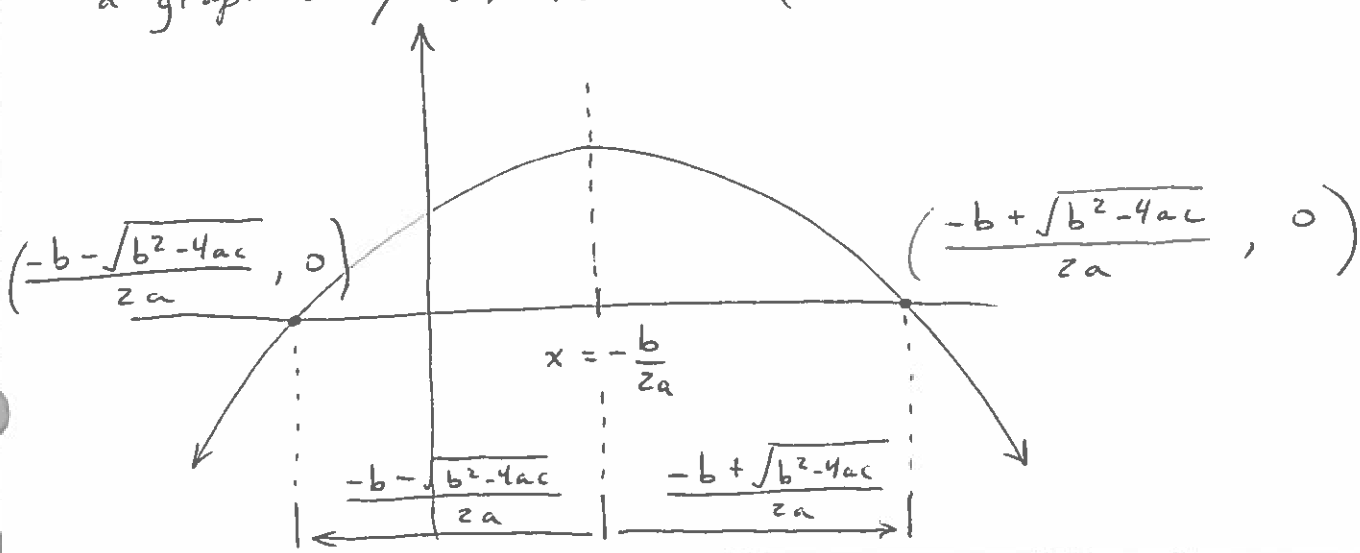
$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x_1=-\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\therefore \boxed{x_1=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}}$$

$$\therefore \boxed{x_2=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}$$

We can illustrate these results by sketching a graph of $y=ax^2+bx+c$ (we will show $a<0$):



Summary: The quadratic formula can be written as $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

This formula takes a quadratic relation in standard form $y = ax^2 + bx + c$

and then completes the square and solves for x by opposite operations when y = 0

to find axis of symmetry $x = -\frac{b}{2a}$ and x-intercepts $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Example – Determine the vertex and x-intercepts of $y = -0.8x^2 + 6x + 5$.

axis of symmetry: $x = -\frac{b}{2a}$

$x = \frac{-6}{2(-0.8)}$

$x = 3.75$

maximum: $y = -0.8(3.75)^2 + 6(3.75) + 5$

$y = 16.25$

vertex: $\boxed{(3.75, 16.25)}$

x-intercepts: $0 = -0.8x^2 + 6x + 5$

$x = \frac{-6 \pm \sqrt{(6)^2 - 4(-0.8)(5)}}{2(-0.8)}$

$x = \frac{-6 \pm \sqrt{52}}{-1.6}$

$\boxed{x_1 \doteq -0.757 \quad \text{or} \quad x_2 \doteq 8.257}$